ELSEVIER

Journal of Geometry and Physics 17 (1995) 231–244

JOURNAL OF GEOMETRY AND PHYSICS

# Holomorphic realization of $\bar{\partial}$-cohomology and constructions of representations *

Michael G. Eastwood [a,1], Simon G. Gindikin [b,*], Hon-Wai Wong [b,2]

[a] *Department of Pure Mathematics, University of Adelaide, South Australia 5005, Australia*

[b] *Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA*

Received 9 February 1995

**Abstract**

A language is developed for $\bar{\partial}$-cohomology, which is different from both the Dolbeault and the Čech descriptions, and involves only holomorphic objects. This language is then illustrated in certain cases of interest to representation theory. This makes possible a new geometric construction of the ladder representations for $SU(2, p)$ and the non-holomorphic discrete series representations of $SU(2, 1)$. The constructions are closely related to Penrose transforms.

*Keywords:* Čech theory; Discrete series; Dolbeault cohomology; Hodge theory; Ladder representations
*1991 MSC:* 22E45, 22E46, 32F10, 32M15

## 0. Introduction

We develop in this paper a holomorphic language for $\bar{\partial}$-cohomology and apply it to representation theory. The outline of this project was discussed in [G2]. The idea is to associate a Stein manifold $X$ to a complex manifold $M$. The $\bar{\partial}$-cohomology of $M$ may then be realized by a purely holomorphic construction on $X$. When $M$ is a flag manifold for a semisimple Lie group $G$, the Stein manifold $X$ may often be chosen explicitly so that $G$ acts

* The first author is supported by an ARC Senior Research Fellowship. The second author is supported by NSF Grant DMS 9202049. Part of the research of this paper was conducted during the second author's visit to the University of Adelaide, also supported by the ARC.

* Corresponding author. E-mail: gindikin@math.rutgers.edu.

[1] E-mail: meastwoo@spam.maths.adelaide.edu.au.

[2] E-mail: hwong@math.rutgers.edu.



*SSDI* 0393-0440(95)00035-6

on this construction. As a result, we can give a holomorphic realization of the associated representations.

It turns out that sometimes there is a 'holomorphic' version of Hodge theory: canonical representatives can be picked from the cohomology classes by means of complex analytic structures only. Of course, we cannot generally expect such a Hodge theory (in the sense just indicated), but it is our belief that the structures involved in representation theory are rich enough for such a theory. It should be mentioned that this Hodge theory depends crucially on the geometry of certain maximally compact submanifolds (called linear cycles) of the flag domains involved. In this paper we consider a few examples to illustrate the program just explained.

All these are closely related to Penrose transform, as is the content of [G2], which in turn follows [GH,G1]; however, the machinery is different. Both [GH,G1] give explicit formulae for Penrose transforms. They use the method of integral transform to map the space of Dolbeault cohomology to the solution space of certain differential equations (which generalize the zero-rest-mass equation) on the space of the linear cycles. They then map this solution space to our space of canonical representatives by means of a differential operator $\kappa$. The papers [RSW,S,WW,W] are of related interest.

It is hoped that in the future we can produce the unitary structures of the representations by considering suitable integrals involving the canonical representatives. In fact, concrete formulae have been proposed in [G2] for that purpose. We believe that these, or suitable variations thereof, should yield the unitary structure we are after.

Here is a brief outline of the paper. In Section 1.1, a Čech theory with holomorphic parameters is developed. Then in Section 1.2, another and more direct way of relating Čech cohomology groups with Dolbeault cohomology groups is explained. This ends the general part of this paper. The second part is devoted to illustrating the aforementioned Hodge theory. Section 2.1 explains the aim of this theory and illustrates it in the case of an arbitrary generalized flag manifold. Concrete formulae are given for the case of $\mathbb{P}_1$; this result is basic to all the subsequent sections. Section 2.2 describes the ladder representations of $SU(2, p)$ and Section 2.3 is on the Hodge theory in the case of the non-holomorphic discrete series representations of $SU(2, 1)$.

## 1. General theory

### 1.1. Čech theory with holomorphic parameters

Let $M$ be a complex manifold and $V \to M$ a holomorphic vector bundle. To compute $H^r(M, \mathcal{O}(\mathcal{V}))$, Čech theory is often employed, usually with a Stein cover. In [G2], a Čech theory with smoothly varying parameters is developed. The idea is as follows. Instead of using a Stein cover $\{U_j\}_{j\in J}$, where $J$ has no special structure, use a Stein cover $\{U_\xi\}_{\xi\in\Xi}$ where $\Xi$ is a manifold and $U_\xi$ depends 'smoothly' on $\xi$. In ordinary Čech theory the differentials are difference operators whereas, as discussed in [G2], in the 'smooth' theory they are differential operators. Here is a variation mentioned in [G2]. The difference is that complex analytic parameters are used, and the existence of an open cover consisting of Stein sets is replaced by a closely related condition. To establish the isomorphism between Čech cohomology groups and the corresponding Dolbeault cohomology groups, a double complex is used, into which both the Dolbeault complex and the Čech complex inject via chain maps; it turns out that both chain maps are quasi-isomorphisms; see [Gu] for details. As a general rule, the results and their proofs in this and the next section are modifications of the corresponding case of Čech cohomologies.

**Theorem 1.1.** *Suppose that $\pi : X \to M$ is a holomorphic submersion with contractible fibers and that $X$ is Stein. Then the holomorphic relative de Rham cohomologies $H^r(\Gamma(X, \Omega^\bullet_\pi(V)))$ are canonically isomorphic to the Dolbeault cohomologies $H^r(M, \mathcal{O}(V))$. Here $\Omega^\bullet_\pi$ denotes the holomorphic relative de Rham complex.*

*Proof.* While the proof of this theorem is standard (see, for example, [BE, pp. 69–72]), for our purposes the following explicit construction is more useful. Firstly, consider the case where $V$ is the trivial line bundle. Let $\Lambda^r_X$ denote the sheaf of germs of smooth complex-valued $r$-forms on $X$. Define the sheaf $A^1$ by

$$A^1 \equiv \Lambda^1_X/\pi^*\Lambda^{1,0}_M \text{ and, more generally, } A^r \equiv \bigwedge^r A^1 = \Lambda^r_X/(\pi^*\Lambda^{1,0}_M \wedge \Lambda^{r-1}_X).$$

The short exact sequence

$$0 \to \pi^*\Lambda^{0,1}_M \to A^1 \to \Lambda^1_\pi \to 0$$

induces a filtration of the complex $\Gamma(X, A^\bullet)$ and thereby a spectral sequence (see [BES] or [W] for details). The contractibility of the fibers implies that this spectral sequence degenerates into an isomorphism

$$H^r(M, \mathcal{O}) \cong H^r(\Gamma(X, A^\bullet)). \tag{1.1}$$

Now we use that $X$ is a complex manifold to decompose $A^r$:

$$A^r = \bigoplus_{p+q=r} A^{p,q}, \quad \text{where } A^{p,q} \equiv \begin{cases} \Lambda^{p,q}_X/(\pi^*\Lambda^{1,0}_M \wedge \Lambda^{p-1,q}_X) & \text{for } p \geq 1, \\ \Lambda^{p,0}_\pi \otimes \Lambda^{0,q}_X & \text{for all } p. \end{cases}$$

This is a double complex whose total complex is just $A^\bullet$. Since $X$ is Stein, the cohomology of the complex

$$\Gamma(X, \Lambda^{p,0}_\pi \otimes \Lambda^{0,\bullet}_X)$$

is $\Gamma(X, \Omega^p_\pi)$ in degree zero and otherwise vanishes. Therefore, the corresponding spectral sequence implies that

$$H^r(\Gamma(X, A^\bullet)) \cong H^r(\Gamma(X, \Omega^\bullet_\pi)). \tag{1.2}$$

Combining (1.1) and (1.2) gives the required natural isomorphisms. The proof for a general $V$ is obtained simply by tensoring with $V$ throughout. Thus, we define

$$A^1(V) \equiv (\Lambda^1_X \otimes \pi^* V)/\pi^*(\Lambda^{1,0} \otimes V)$$

and so on. It is easy to check that this does not affect the argument. □

**Remark 1.2.** Each of the complexes (i.e. the holomorphic relative de Rham complex, the Dolbeault complex, and the complex $A^\bullet$) has a natural Fréchet topology, and the two quasi-isomorphic chain maps (each injects into $A^\bullet$) are continuous. It is well known that a continuous quasi-isomorphism between complexes of Fréchet spaces induces a quasi-isomorphism of topological vector spaces. Therefore, computing sheaf cohomologies using either the holomorphic de Rham complex or the Dolbeault complex yields the same topological vector spaces. In particular, if one complex has the closed range property in a particular degree, then so does the other.

### *1.2. Pulling-back the Čech complex*

The isomorphism of Section 1.1 can be obtained directly as follows. Choose a smooth section $\gamma : M \to X$ of $\pi : X \to M$. Since the fibers of $\pi$ are contractible, this is always possible (see [St, Sections 29.8 and 34.2]) and, in case $X$ is a Stein neighborhood of $M$ inside its complexification, we may take $\gamma$ to be the tautological diagonal embedding. For $\omega \in \Gamma(X, \Omega^r_\pi(V))$, locally we may choose $\widetilde{\omega}$ a section of $\Lambda^{r,0}(V)$, such that $\widetilde{\omega} \mapsto \omega$ under the natural projection

$$\Lambda^{r,0}(V) \to \Lambda^{r,0}_\pi(V) \supset \Omega^r_\pi(V).$$

We may now form $(\gamma^*\widetilde{\omega})^{0,r}$, the $(0,r)$-component of the pull-back of $\widetilde{\omega}$ to $M$ under $\gamma$. In fact, this is independent of choice of $\widetilde{\omega}$ since, if $\widetilde{\widetilde{\omega}}$ is another, then we may write

$$\widetilde{\omega} - \widetilde{\widetilde{\omega}} = \sum_{i=1}^{N} \pi^*\alpha_i \wedge \beta_i$$

for suitably chosen $(1,0)$-forms $\alpha_i$ on $M$ with coefficients in $V$ and $(r-1,0)$-forms $\beta_i$ on $X$. Since $\pi \circ \gamma = 1_M$, we conclude that

$$\gamma^*\widetilde{\omega} - \gamma^*\widetilde{\widetilde{\omega}} = \sum_{i=1}^{N} \alpha_i \wedge \gamma^*\beta_i$$

has no $(0,r)$-component. Let $\kappa\omega \in \Gamma(M, \Lambda(V))$ be defined by the local formula $\kappa\omega = (\gamma^*\widetilde{\omega})^{0,r}$.

**Theorem 1.3.** *The definition of $\kappa$ as above gives a chain mapping*

$$\kappa : \Gamma(X, \Omega^\bullet_\pi(V)) \to \Gamma(M, \Lambda^{0,\bullet}(V)),$$

*which induces the isomorphism of Theorem* 1.1.

*Proof.* Firstly, consider the case where $V$ is the trivial line bundle. To see that $\kappa$ is a chain mapping, notice that we may take $\widetilde{\omega}$ to be holomorphic, i.e. a local section of $\Omega^r$. We may then choose $\widetilde{d\omega} = d\widetilde{\omega}$ whence

$$\kappa d\omega = (\gamma^* \widetilde{d\omega})^{0,r+1} = (\gamma^* d\widetilde{\omega})^{0,r+1} = (d\gamma^* \widetilde{\omega})^{0,r+1} = \bar{\partial}((\gamma^* \widetilde{\omega})^{0,r}) = \bar{\partial}\kappa\omega$$

and so $\kappa$ is a chain map. A careful examination of the proof of Theorem 1.1 shows that if $\omega \in \Gamma(X, \Omega_\pi^r)$ satisfies $d_\pi \omega = 0$, then the corresponding $\tau \in \Gamma(M, \Lambda^{0,r})$ is characterized up to cohomological freedom by the equation

$$\widetilde{\omega} - \pi^* \tau + \rho = d\sigma,$$

where $\sigma \in \Gamma(X, \Lambda^{r-1}), \widetilde{\omega} \in \Gamma(X, \Lambda^{r,0})$ is a lift of $\omega \in \Gamma(X, \Lambda_\pi^{r,0}), \tau \in \Gamma(M, \Lambda^{0,r})$, $\rho \in \Gamma(X, \pi^* \Lambda^{1,0} \wedge \Lambda^{r-1})$.

Since $\pi \circ \gamma = 1_M$, it follows that $(\gamma^* \rho)^{0,r} = 0$. Hence,

$$\kappa\omega - \tau = (\gamma^* \widetilde{\omega})^{0,r} - \tau = (\gamma^* d\sigma)^{0,r} = (d\gamma^* \sigma)^{0,r} = \bar{\partial}((\gamma^* \sigma)^{0,r-1})$$

and so $\kappa\omega$ and $\tau$ represent the same cohomology class, as required. For a general $V$, the same argument goes through except that, in order to differentiate forms with coefficients in $V$, one needs to choose a connection on $V$ compatible with the complex structure. □

**Remark 1.4.** The definition of $\kappa$ involves choosing a smooth section $\gamma$ of the holomorphic submersion $\pi : X \to M$. In making this choice we are leaving the holomorphic category but this is to be expected and is entirely consistent with our philosophy, since the definition of Dolbeault cohomology also involves non-holomorphic objects; the link between the holomorphic and non-holomorphic categories is this non-holomorphic section $\gamma$.

## 2. Illustrating examples

### *2.1. Hodge theory: Aim and first example*

By a holomorphic Hodge theory is meant the study of the following situation. Represent a Dolbeault cohomology group as a holomorphic relative de Rham cohomology group (as in Section 1.1); sometimes, as will be seen, there is a preferred representative in each relative de Rham class. This representative, which, for lack of better terminology, is called a holomorphic harmonic form, is picked out as a solution of a differential operator which decreases the degree of the relative differential form by one. When it occurs, this is clearly analogous to the ordinary Hodge decomposition. This decomposition pertains in many interesting settings (and for non-compact manifolds); the second part of this paper is devoted to various examples which illustrate this phenomenon.

The first example is that of an arbitrary generalized flag manifold with a homogeneous vector bundle on it. For that purpose, note that a generalized flag manifold can be thought of as $K/L$, where $K$ is a compact Lie group, and $L$ the centralizer of a torus. As a complex manifold, $M = K_{\mathbb{C}}/Q$, where $Q$ is a parabolic subgroup of $K_{\mathbb{C}}$, and there is a Levi decomposition $Q = L_{\mathbb{C}}U$. Let $X = K_{\mathbb{C}}/L_{\mathbb{C}}$, by a theorem of Matsushima [Ma], $X$ is Stein. The obvious projection $\pi : X \to M$ gives a $K_{\mathbb{C}}$ homogeneous holomorphic fiber bundle and the typical fiber is $U$, which is Euclidean. So the situation in Section 1 is obtained. In fact, there is a $K$-equivariant section $\gamma : M \to X$, which sends $k \cdot L$ to $k \cdot L_{\mathbb{C}}$. It may be of interest to observe that $X$ is an open subset of $K_{\mathbb{C}}/Q \times K_{\mathbb{C}}/\overline{Q}$, where $\overline{Q}$ is the complex conjugate of $Q$, viewed as a complex subgroup of $K_{\mathbb{C}}$.

Denote by $\overline{U}$ the complex conjugate of $U$ and by $\overline{\mathfrak{u}}$ and $\mathfrak{u}$ the corresponding Lie algebras. They may, depending on the context, denote *complexified* Lie algebras or *real* Lie algebras of complex groups. Now let $V$ be a finite-dimensional irreducible $Q$-module, this induces a holomorphic vector bundle $V \to M$.

The holomorphic relative de Rham complex $\Gamma(M, \Omega^{\bullet}_{\pi}(V))$ can be represented, via pull-back from $X$ to $K_{\mathbb{C}}$, as $C^i \equiv (\mathcal{O}(K_{\mathbb{C}}) \otimes V \otimes \wedge^i \mathfrak{u}^*)^{L_{\mathbb{C}}}$. The relative differential is given by $d_\pi \equiv -\{r(X_i) \otimes e(\overline{X}_i) + \frac{1}{2} I \otimes e(\overline{X}_i) \circ \mathrm{ad}(X_i)\}$ (where summation over the subscripts is assumed). Here $\{X_1, X_2, \ldots\} \subseteq \mathfrak{u}$ is a basis chosen so that $(X_i, \overline{X}_j) = -\delta_{ij}$, (where $(\ ,\ )$ is the Killing form of $\mathfrak{k}_{\mathbb{C}}$), $e(\ )$ means exterior product, and $\overline{u}$ is identified with $\mathfrak{u}^*$ via the Killing form.

Next, define the dual operator $d^*_\pi : C^i \to C^{i-1}$ by

$$d^*_\pi \equiv -\{r(\overline{X}_i) \otimes \iota(X_i) + \tfrac{1}{2} I \otimes \mathrm{ad}(\overline{X}_i) \circ \iota(X_i)\}.$$

Again, summation over the subscripts is assumed and the notation $\iota$ stands for contraction. Observe that, for each $i$, we have $\mathrm{ad}(\overline{X}_i) \circ \iota(X_i) = \iota(X_i) \circ \mathrm{ad}(\overline{X}_i)$. The definition of this dual operator is motivated by the following lemma.

**Lemma 2.1.** $\kappa d^*_\pi = \overline{\partial}^* \kappa$.

*Proof.* Use [GS, 5.7]. The Cauchy–Riemann equations are implicitly used. □

**Remark 2.2.**

(1) The action of $\mathrm{ad}(\overline{X}_i)$ on $\mathfrak{u}^*$ is understood as follows. Extend an element of $\mathfrak{u}^*$ to one of $\mathfrak{g}^*$ by zero on $\mathfrak{l}_{\mathbb{C}} \oplus \overline{\mathfrak{u}}$; as such, $\mathrm{ad}(\overline{X}_i)$ makes sense.

(2) The adjoint Dolbeault operator $\overline{\partial}^*$ is defined relative to the Hermitian metric defined by using the Killing form of $\mathfrak{k}$.

(3) By an elementary computation, $d^*_\pi = -r(\overline{X_i}) \otimes \iota(X_i)$ when the form is of top degree. (Here, of course, the summation convention is in place.)

**Lemma 2.3.** *For each class in $H^r(C^\bullet)$, there is at most one* harmonic *representative, i.e. a holomorphic relative $r$-form on $X$ for which $d_\pi \omega = 0 = d^*_\pi \omega$.*

*Proof.* Suppose $\omega_1$ and $\omega_2$ are cohomologous and harmonic. Let $\phi = \omega_1 - \omega_2$. Then $d_\pi \phi = 0 = d^*_\pi \phi$ and $\phi = d_\pi \psi$ for some $\psi$.

Apply $\kappa$ to all the equations above and recall Lemma 2.1. It follows that $\kappa\phi$ is a harmonic form in the usual sense and is exact. Hence $\kappa\phi = 0$ and therefore $\gamma^*\phi = 0$. However, the image of $\gamma$ sits inside $X$ as a totally real submanifold, so $\phi = 0$. □

From this immediately follows the corollary.

**Corollary 2.4.** *The space of holomorphic harmonic forms injects naturally into the space of relative cohomology.*

To show that there *exists* a harmonic representative, the assumption that $V$ is irreducible becomes crucial. It is further assumed that the cohomology is of the *top* degree, i.e. in degree $s = \dim_{\mathbb{C}} M$.

The strategy is to construct explicitly some (non-zero) holomorphic harmonic forms; then, by the irreducibility of the Dolbeault cohomology space in question (using the generalized Borel–Weil–Bott theorem), the existence is proved. In order to make sense of the construction, the following observation is helpful.

**Lemma 2.5.** *Either*

(1) *the $Q$-module $V$ is of the form $W^{\#} \equiv (W^{*[\bar{\mathfrak{u}}]})^* \otimes \mathbb{C}_{-2\rho(\bar{\mathfrak{u}})}$, where $W$ is an irreducible $K$-module and $[\bar{\mathfrak{u}}]$ denotes "the $\bar{\mathfrak{u}}$ invariants of", or*

(2) $H^s(M, V) = 0$.

*Proof.* Without loss of generality, let $W$ have the lowest weight (with respect to a chosen maximal torus and positive root system) $\lambda + 2\rho_K$. Thus, $W^*$ is irreducible with highest weight $-(\lambda + 2\rho_K)$, and hence $W^{*[\bar{\mathfrak{u}}]}$ is an irreducible $L$-module with highest weight $-(\lambda + 2\rho_K)$ (with respect to the same maximal torus and the compatible system of positive roots of $L$). Therefore, $W^{\#}$ is irreducible with the lowest weight $\lambda + 2\rho_L$.

Any irreducible $L$-module $V$ has lowest weight of the form $\lambda + 2\rho_L$. The question is therefore: for an $L$-anti-dominant integral weight $\lambda + 2\rho_L$, when does $\lambda + 2\rho_K$ turn out to be $K$-anti-dominant?

To answer this question, observe that $(\rho(\bar{\mathfrak{u}}), \alpha) = 0$ for all simple roots of $L$, so $(\lambda + 2\rho_K, \alpha) \leq 0$ for all positive roots of $L$. Hence $\lambda + 2\rho_K$ is not anti-dominant precisely when $(\lambda + 2\rho_K, \alpha) > 0$ for some $\alpha \in \Delta(\bar{\mathfrak{u}})$.

Now examine the fibration $K/T \to K/L$, where $T$ is the maximal torus chosen throughout and consider the associated Leray spectral sequence for the cohomologies of the line bundle $\mathcal{L}_\lambda$ induced from the character $\mathbb{C}_\lambda$. Its $E_2$ terms are given by

$$E_2^{p,q} = H^q(K/L, H^p(L/T, \mathcal{L}_\lambda)).$$

By the anti-dominance of $\lambda$, it follows that $E_2^{p,q} = 0$ except when $p = l$, the (complex) dimension of $L/T$. Further, $E_2^{l,q} = H^q(K/L, W^{\#})$.

It is clear from the Borel–Weil–Bott theorem that $H^q(K/T, \mathcal{L}_\lambda) = 0$ when $q$ is the top degree, given the assumption that $(\lambda + 2\rho_K, \alpha) > 0$ for some $\alpha \in \Delta(\bar{\mathfrak{u}})$. This completes the proof. □

By Lemma 2.5, to prove the existence of harmonic representatives, it can be assumed that $V = W^{\#} \equiv (W^{*[\bar{\mathfrak{u}}]})^* \otimes \mathbb{C}_{-2\rho(\bar{\mathfrak{u}})}$. Pick a basis $\{\lambda_1, \ldots, \lambda_k\}$ of $W^{*[\bar{\mathfrak{u}}]}$ and let $\{e_1, \ldots, e_k\}$ be the dual basis; denote by $\langle\ ,\ \rangle$ the pairing. Finally, pick a basis $\{\omega^{-\alpha_1}, \ldots, \omega^{-\alpha_s}\}$ for $\mathfrak{u}^*$ and let $\omega^{-\alpha} \equiv \omega^{-\alpha_1} \wedge \cdots \wedge \omega^{-\alpha_s}$.

For any $w \in W$ define $\phi_w : K_{\mathbb{C}} \to W^{\#}$ by $\phi_w(k) = (\langle k^{-1}w, \lambda_i\rangle \otimes \omega^{-\alpha}) \otimes (e_i \otimes 1_{-2\rho(\bar{\mathfrak{u}})})$. It is routine to verify that the definition is independent from the choice of $\{\lambda_i\}$ and that $\phi_w$ defines a holomorphic relative top form on $K_{\mathbb{C}}/L_{\mathbb{C}}$ with values in the vector bundle induced from $W^{\#}$.

Clearly $\phi_w$ is non-zero and closed; it remains to see that it is harmonic. By Remark 2.2, it suffices to show $r(\overline{X_i})\phi_w = 0$. However, $r(\overline{X_i})\phi_w(k) = -\langle k^{-1}v, \overline{X_i}\lambda_j\rangle \otimes (e_j \otimes 1_{-2\rho(\bar{\mathfrak{u}})})$ and since $\lambda_j \in W^{*[\bar{\mathfrak{u}}]}$, this expression vanishes.

The following proposition summarizes all these arguments.

**Proposition 2.6.** *When $V$ is irreducible and $H^s(M, V) \neq 0$, there is a non-zero holomorphic harmonic form.*

By the generalized Borel–Weil–Bott theorem, it is known that, under the conditions of Proposition 2.6, $H^s(M, V)$ is an *irreducible* representation of $K$. The space of all holomorphic harmonic $s$-forms also forms a representation. From Corollary 2.4 and Proposition 2.6, we may deduce that these representations agree. We have proved the following corollary.

**Corollary 2.7.** *When $V$ is irreducible, there is a unique holomorphic harmonic representative in each holomorphic relative de Rham class of degree $s$.*

Here is an equivalent, but sometimes more useful, statement.

**Corollary 2.8.** *There is a direct sum decomposition*

$$C^s = \ker d_\pi^* \oplus \operatorname{im} d_\pi .$$

The rest of this section is devoted to the case of $M = \mathbb{P}_1$. This is, of course, a special case of what has just been described. It is included here because the formulae are so simple, and also because this is the case which will be used in the following sections. See [ET] for notational details.

Let

$$X = \mathbb{P}_1 \times \mathbb{P}_1^* \setminus \{([z_i], [\xi^i]) \text{ s.t. } z_i\xi^i = 0\},$$

with $\pi : X \to M$ being projection onto the first factor. We are using here the Einstein summation convention or, more precisely, Penrose's abstract index notation [PR]. The smooth section $\gamma : M \to X$ is given by $\gamma([z_i]) = ([z_i], [\overline{z}^i])$.

Let $\mathcal{O}(k)$ denote the usual holomorphic line bundle on $\mathbb{P}_1$ whose sections can be identified locally as holomorphic functions $f(z_i)$ homogeneous of degree $k$. Let $\Omega_\pi^\bullet(k)$ denote the holomorphic relative de Rham complex on $X$ with coefficients in the pull-back of $\mathcal{O}(k)$ to $X$. Let $\mathcal{O}(k,l)$ denote the sheaf of germs of holomorphic functions of $(z_i, \xi^i)$ homogeneous of degree $(k,l)$ in the sense that $f(\lambda z_i, \mu\xi^i) = \lambda^k\mu^l f(z_i, \xi^i)$. This is the sheaf of local sections of a holomorphic line bundle on $X$.

The bundle $\Omega_\pi^1(k)$ may be identified with $\mathcal{O}(k, -2)$ in which case the inclusion into $\mathcal{O}_i(k-1, -1)$ is given by $g \mapsto \epsilon_{ij}\xi^j g$. Therefore, the operator $d_\pi : \mathcal{O}(k,0) \to \mathcal{O}(k, -2)$

may be given directly by the formula

$$f \mapsto \frac{1}{z_m \xi^m} \epsilon^{ij} z_j \frac{\partial f}{\partial \xi^i}.$$

The formula for the 'adjoint' is

$$\begin{array}{ccc} \mathcal{O}(k,-2) & \xrightarrow{d_\pi^*} & \mathcal{O}(k,0) \\ \Cup & & \Cup \\ g & \longmapsto & z_m \xi^m \epsilon_{ij} \xi^j \dfrac{\partial g}{\partial z_i} \end{array}$$

The operators are the same, up to a non-zero multiple, as the ones given in the context of a generalized flag manifold.

### 2.2. *Application to certain ladder representations*

Let $G = \mathrm{SU}(2,p)$. On the domain

$$M = \{[z_I] \in \mathbb{P}_{p+1} \text{ s.t. } |z_1|^2 + |z_2|^2 - |z_3|^2 - \cdots - |z_{p+1}|^2 - |z_{p+2}|^2 > 0\},$$

the group $G$ acts transitively. (We are using upper case abstract indices, in effect ranging over $\{1, 2, \ldots, p+2\}$, to distinguish from the previous section. In the next section we shall have need for both types.) In fact $M = G/L$, where $L = \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1,p))$. Let $\mathcal{O}(k)$ denote the usual holomorphic line bundle on complex projective space and hence on $M$. Let

$$X = \left\{ (z,\xi) \in M \times M \text{ s.t. } \begin{array}{l} z \neq \xi \text{ and the line joining } z \text{ and } \xi \text{ in projective} \\ \text{space lies entirely inside } M \end{array} \right\}$$

and

$$Y = \left\{ l \in \mathrm{Gr}_2(\mathbb{C}^{p+2}) \text{ s.t. } \begin{array}{l} \text{when } l \text{ is regarded as a line in } \mathbb{P}_{p+1} \\ \text{it lies entirely inside } M. \end{array} \right\}.$$

Note that $Y = G/K$ where $K = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(p))$. It is well known that $Y$ is Stein (it may be realized as a tube domain over a cone). We maintain that $X$ is also Stein. To see this, define a mapping

$$\begin{array}{ccc} X & \longrightarrow & (\mathbb{P}_1 \times \mathbb{P}_1 \setminus \text{diagonal}) \times Y \\ \Cup & & \Cup \\ ([z_I],[\xi_I]) & \longmapsto & (([z_1,z_2],[\xi_1,\xi_2]), \text{the line joining } z \text{ and } \xi) \end{array} \tag{2.1}$$

It is easy to check that this is a biholomorphism. The right-hand side is a product of Stein manifolds and hence Stein.

Define $\pi : X \to M$ as projection onto the first factor. The fibers are homeomorphic to the Cartesian product of $\mathbb{C}$ with a ball in $\mathbb{C}^p$. In particular, they are contractible. From Theorem 1.1, we may conclude that $H^r(\Gamma(X, \Omega_\pi^\bullet(k)))$ is isomorphic to the Dolbeault cohomology $H^r(M, \mathcal{O}(k))$. In order to study $H^r(\Gamma(X, \Omega^\bullet_\pi(k)))$, let us introduce one more space

$$F = \{(z,l) \in M \times Y \text{ s.t. } z \in l\}$$

'between' $X$ and $M$ in that $\pi$ factors as

$$X \xrightarrow{\beta} F \xrightarrow{\alpha} M.$$

Here, $\beta(z,\xi) = (z,l)$ where $l$ is the line joining $z$ and $\xi$, and $\alpha(z,\xi) = z$. The short exact sequence

$$0 \to \beta^*\Omega^1_\alpha(k) \to \Omega^1_\pi(k) \to \Omega^1_\beta(k) \to 0$$

induces a filtration of the complex $\Gamma(X, \Omega^\bullet_\pi(k))$ and hence a spectral sequence

$$E_0^{p,q} = \Gamma(X, \Omega^q_\beta \otimes \beta^*\Omega^p_\alpha(k)) \Longrightarrow H^{p+q}(\Gamma(X, \Omega^\bullet_\pi)).$$

For fixed $p$, the differentials at this level are the relative exterior derivatives $d_\beta$. The fibers of $\beta$ are homeomorphic to $\mathbb{C}$ and we may apply Theorem 1.1 again to conclude that

$$E_1^{p,q} = H^q(F, \Omega^p_\alpha(k)). \tag{2.2}$$

Indeed, more precisely, the mapping

$$\begin{array}{ccc} F & \longrightarrow & \mathbb{P}_1 \times Y \\ \Cup & & \Cup \\ (z,l) & \longmapsto & ([z_1, z_2], l) \end{array}$$

is a biholomorphism compatible with (2.1), so we may regard $\beta : X \to F$ as simply the Cartesian product with $Y$ of the situation discussed in Section 2.1. In particular, we may deduce from Corollary 2.8, that there is a direct sum decomposition

$$\begin{aligned} \Gamma(X, \Omega_\beta(k)) = &\ker d^*_\beta : \Gamma(X, \Omega^1_\beta(k)) \to \Gamma(X, \mathcal{O}(k)) \\ &\oplus \operatorname{im} d_\beta : \Gamma(X, \mathcal{O}(k)) \to \Gamma(X, \Omega^1_\beta(k)). \end{aligned} \tag{2.3}$$

Here, if we identify $\Omega^1_\beta(k)$ as $\mathcal{O}(k+1, -1)$, then $d^*_\beta : \Omega^1_\beta(k) \to \Omega^0_\beta(k) = \mathcal{O}(k)$ is given by

$$d^*_\beta g = \xi_I \frac{\partial g}{\partial z_I}.$$

Define $d^*_\pi : \Omega^1_\pi(k) \to \Omega^0_\pi(k)$ to be the natural projection $\Omega^1_\pi(k) \to \Omega^1_\beta(k)$ followed by $d^*_\beta$.

**Theorem 2.9.** *For $k \leq -2$, each relative de Rham class in $H^1(\Gamma(X, \Omega^\bullet_\pi(k)))$ has a unique representative $\omega$ such that $d^*_\pi \omega = 0$.*

*Proof.* According to (2.3), every relative de Rham class can be represented by a relative form $\omega$ on $X$ with $d^*_\pi \omega = 0$. Suppose $\widetilde{\omega}$ is another such form in the same cohomology class, i.e.

$$\omega - \widetilde{\omega} = d_\pi f$$

for some $f \in \Gamma(X, \mathcal{O}(k))$. Let $\nu = \omega - \widetilde{\omega}$. Then $\nu \in \Gamma(X, \Omega^1_\pi(k))$ satisfies $d^*_\pi \nu = 0$ and is of the form $d_\pi f$. Its image $\overline{\nu}$ in $\Gamma(X, \Omega^1_\beta(k))$ satisfies $d^*_\beta \overline{\nu} = 0$ and is of the form $d_\beta f$. From (2.3), it follows that $\overline{\nu} = 0$. In other words, $\nu \in \Gamma(X, \beta^* \Omega^1_\alpha(k))$. Now, in particular, $d_\beta \nu \in \Gamma(X, \Omega^1_\beta \otimes \beta^* \Omega^1_\alpha(k))$ vanishes and so $\nu = \beta^* \mu$ for some $\mu \in \Gamma(F, \Omega^1_\alpha(k))$. Recall that $F \cong \mathbb{P}_1 \times Y$. It is easy to check that, when restricted to each $\mathbb{P}_1$,

$$\Omega^1_\alpha(k) \cong \underbrace{\mathcal{O}(k+1) \oplus \mathcal{O}(k+1) \oplus \cdots \oplus \mathcal{O}(k+1)}_{p}$$

(see, for example, [E] where $\Omega^1_\alpha(k)$ is denoted $(-k|1|-1, 0, \ldots, 0)$). If $k \leq -2$, this cannot have any global sections. Hence $\mu = 0$, and uniqueness is shown. □

**Remark 2.10.** Since the relative de Rham cohomologies are isomorphic to the corresponding Dolbeault cohomologies, by [Wo], these spaces of holomorphic harmonic forms are the maximal globalizations of the underlying Harish–Chandra modules (which are, in fact, Zuckerman modules). The condition $k \leq -p-1$ is the weakly good condition in [V, Theorem $A_1$] for the Zuckerman modules. In fact, for vanishing of cohomologies of degree other than one, $k \leq -1$ is sufficient; see [RSW, 10.4]. The proof of Theorem 2.9 also shows this vanishing for $k \leq -1$. Note that the spectral sequence (2.2) is the usual spectral sequence (e.g. [BE, p.72]) for the Penrose transform.

In [G2], a different harmonic condition is used. For a closed relative one-form $\omega = \omega^I(z, \xi) d\xi_I$, our harmonic condition is precisely $\xi_J(\partial/\partial z_J)(\omega^I z_I) = 0$, whereas the condition in [G2], henceforth called "the strong harmonic condition", is $\xi_J(\partial/\partial z_J)\omega^I = 0$ for all $I$. Clearly, the strong harmonic condition implies the harmonic condition. The converse is true when the Dolbeault cohomology space of degree one is irreducible (or zero) as a $G$-module (for example, if the weakly good condition $k \leq -p-1$ holds, see [V, Theorem $A_1$]). To see this, observe that the space of strongly harmonic one-forms is a $G$-invariant submodule, so it suffices to see that it is non-zero. The form

$$\omega = \frac{\xi_1^{-k-2}}{(z_0\xi_1 - z_1\xi_0)^{-k}} (\xi_0 d\xi_1 - \xi_1 d\xi_0)$$

is easily verified to be strongly harmonic (and is well defined if $k \leq -2$). Hence, we have the following lemma.

**Lemma 2.11.** *When $k \leq -p-1$ our harmonic condition is equivalent to the* (*strongly*) *harmonic condition in* [G2].

### 2.3. *Discrete series of* SU(2, 1)

Let $G = \mathrm{SU}(2,1)$ and $K = \mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$ a maximal compact subgroup. The flag manifold of $G$ is

$$\mathbb{F} = \{(z, Z) \in \mathbb{P}_2 \times \mathbb{P}_2^* \text{ s.t. the point } z \text{ lies on the line } Z\}.$$

The natural $G$ action on $\mathbb{F}$ has three open orbits one of which, say $M$, is associated to the non-holomorphic discrete series. To describe $M$, introduce

$$B \equiv \{[z_I] \in \mathbb{P}_2 \text{ s.t. } |z_1|^2 + |z_2|^2 - |z_3|^2 < 0\},$$

then we have

$$M = \left\{(z, Z) \in \mathbb{F} \text{ s.t. } \begin{array}{l}\text{the point } z \text{ lies outside the closed ball } \overline{B} \\ \text{and the line } Z \text{ intersects } B\end{array}\right\}.$$

It can be verified that $M = G/T$, where $T$ is the diagonal subgroup (i.e. a compact Cartan subgroup). Let

$$X = \left\{(z, Z; u, U) \in M \times M \text{ s.t. } \begin{array}{l} z \neq u \text{ and } Z \neq U, \\ \text{the line joining } z \text{ and } u \text{ lies} \\ \text{outside } \overline{B}, \text{ and the lines } Z \text{ and } U \\ \text{intersect inside } B \end{array}\right\}$$

and

$$Y = \left\{(w, W) \in \mathbb{P}_2 \times \mathbb{P}_2^* \text{ s.t. } \begin{array}{l}\text{the point } w \text{ lies in } B \text{ and} \\ \text{the line } W \text{ lies outside } \overline{B}\end{array}\right\}.$$

It is easy to see that $Y$ is biholomorphic to $B \times B$, and hence Stein. We would like to argue that $X$ is also Stein. To see this, let $L$ be the line in $\mathbb{P}_2$ defined by $z_3 = 0$, and define the map

$$\begin{array}{ccc} X & \longrightarrow & (L \times L \setminus \text{diagonal}) \times Y \\ \Cup & & \Cup \\ (z, Z; u, U) & \longmapsto & (a, b; w, W) \end{array} \tag{2.4}$$

Here, $w$ is the point of intersection of $Z$ and $U$ and $W$ the line joining $z$ and $u$. Also, $a$ is the point of intersection of $Z$ and $L$ and $b$ is where $U$ and $L$ intersect.

It is easy to see that it is a biholomorphism. As before, since this is a product of two Stein manifolds, it is also Stein.

**Remark 2.12.** Points $(w, W)$ of $Y$ parametrize the space of all those $G_{\mathbb{C}}$ translates of the maximal compact subvariety $\mathbb{P}_1 \cong K/T \subseteq G/T$ which remain within $M$; these translates are called *linear cycles*. Given $(w, W)$, the corresponding linear cycle $S(w, W)$ is given explicitly by

$$S(w, W) \equiv \left\{(z, Z) \in M \text{ s.t. } \begin{array}{l}\text{the point } z \text{ lies on the line } W \\ \text{and the line } Z \text{ passes through the point } w\end{array}\right\}.$$

Now define $\pi : X \to M$ to be projection onto the first factor. It is easy to see that it is a fiber bundle with fibers $\Delta \times \Delta \times \mathbb{C}$, where $\Delta$ is the unit disc in $\mathbb{C}$; hence, the fibers are contractible.

Next, we introduce the line bundle. Consider the character $\chi$ of $T$ which sends $\operatorname{diag}(e^{i\theta}, e^{i\phi}, e^{-i(\theta+\phi)})$ to $e^{i(a\theta+b\phi)}$ ($a, b$ are integers). This induces a homogeneous line bundle $\mathcal{O}(r,t)$ of "bidegree" $(r,t)$, where $r=-a, t=b$. To understand the meaning of this "bidegree", observe that $\mathbb{F}$ is a closed submanifold of $\mathbb{P}_2 \times \mathbb{P}_2^*$. Let $\mathcal{O}(r,t)$ be the restriction to $\mathbb{F}$ of the tensor product of a line bundle of degree $r$ over the first factor with one with degree $t$ over the second factor.

We can now follow, almost verbatim, the constructions and arguments in Section 2.2. Some details are omitted.

We use Theorem 1.1, hence $H^r(\Gamma(X,\Omega_\pi^\bullet(r,t)))$ is isomorphic to the Dolbeault cohomology $H^r(M,\mathcal{O}(r,t))$. To proceed further, introduce

$$F \equiv \{(z,Z;w,W) \in M \times Y \text{ s.t. } (z,Z) \in S(w,W)\}.$$

We also have the maps $\pi=\alpha\circ\beta$ as before. Here, $\beta : X \to F$ is the map given by $\beta(z,Z;u,U)=(z,Z;w,W)$, where $w$ is the point of intersection of $Z$ and $U$ and $W$ the line joining $z$ and $u$. Projection onto the first factor induces $\alpha : F \to M$.

Observe that the restriction of $\alpha^*\mathcal{O}(r,t)$ to each fiber of the obvious projection $F \to Y$ is the line bundle of degree $d \equiv r+t=b-a$ over $\mathbb{P}_1$.

Finally, since $F \cong \mathbb{P}_1 \times Y$, we may use Corollay 2.8 to conclude that there is a direct sum decomposition

$$\Gamma(X,\Omega_\beta^1(d)) = \ker d_\beta^* \oplus \operatorname{im} d_\beta.$$

Therefore, we can define $d_\pi^* : \Omega_\pi^1(r,t) \to \Omega_\pi^0(r,t)$ by composing the projection $\Omega_\pi^1(r,t) \to \Omega_\beta^1(r,t)$ with $d_\beta^*$.

**Theorem 2.13.** *For $d=r+t \le -2$, each relative de Rham class in $H^1(\Gamma(X,\Omega_\pi^\bullet(r,t)))$ has a unique form $\omega$ s.t. $d_\pi^*\omega=0$.*

*Proof.* One can copy almost verbatim from that for Theorem 2.9. Replace the degree $k$ by bidegree $(r,t)$. The only new ingredient is that, when restricted to each copy of $\mathbb{P}_1$ (fiber of $F \to Y$), we have

$$\Omega_\alpha^1(r,t) \cong \mathcal{O}(d+1) \oplus \mathcal{O}(d+1).$$

This can be seen as follows. Over the fiber above the point of $Y$ given by $w=[1:0:0]$ and $W=[1:0:0]$, it can be directly verified. This conclusion can be translated to other linear cycles, i.e. to other points of $Y$, by acting with a suitable element of $G_{\mathbb{C}}$. □

**Remark 2.14.** The weakly good condition $(\chi+\rho,\alpha) \le 0$ (cf. [V, Theorem $A_1$]) amounts to $a \ge 1$ and $b \le -1$. This condition guarantees the degree one Dolbeault cohomology to be irreducible (or zero) and unitarizable. Note also that the weakly good condition guarantees the hypothesis for the theorem just stated.

## References

[BES] T.N. Bailey, M.G. Eastwood and M.A. Singer, The Penrose transform for non-holomorphic correspondences, preprint.

[BE] R.J. Baston and M.G. Eastwood, *The Penrose Transform: Its Interaction with Representation Theory* (Oxford University Press, Oxford, 1989).

[E] M.G. Eastwood, The generalised Penrose–Ward transform, Math. Proc. Camb. Phil. Soc. 97 (1985) 165–187.

[ET] M.G. Eastwood and K.P. Tod, Edth—a differential operator on the sphere, Math. Proc. Camb. Phil. Soc. 92 (1982) 317–320.

[G1] S.G. Gindikin, Integral formulas and integral geometry for $\bar{\partial}$-cohomologies in $\mathbb{CP}^n$, Funk. Anal. Prilozh. 18(2) (1984) 26–33 [Engl. transl.: Func. Anal. Appl. 18 (1984) 108–118].

[G2] S.G. Gindikin, Holomorphic language for $\bar{\partial}$-cohomology and representations of real semisimple Lie groups, Contemporary Math. 154 (1993) 103–115.

[GH] S.G. Gindikin and G.M. Henkin, Integral geometry for $\bar{\partial}$-cohomology in $q$-linear concave domains in $\mathbb{CP}^n$, Funk. Anal. Prilozh. 12(4) (1978) 6–23 [Engl. transl.: Func. Anal. Appl. 12 (1978) 247–261].

[GS] P. Griffiths and W. Schmid, Locally homogeneous complex manifolds, Acta Math. 123 (1969) 253–302.

[Gu] R.C. Gunning, *Introduction to Holomorphic Functions of Several Variables* (Wadsworth and Brooks/Cole, 1990).

[Ma] Y. Matsushima, Espaces Homogènes de Stein des Groupes de Lie Complexes I, Nagoya Math. J. 16 (1960) 205–218.

[PR] R. Penrose and W. Rindler, *Spinors and Space-time*, Vol. 1 (Cambridge University Press, Cambridge, 1984).

[RSW] J. Rawnsley, W. Schmid and J. Wolf, Singular unitary representations and indefinite harmonic theory, J. Func. Anal. 51 (1983) 1–114.

[S] W. Schmid, Homogeneous complex manifolds and representations of semisimple Lie groups, Ph.D. Dissertation, Univ. California, Berkeley (1967); Reprinted in *Representation Theory and Harmonic Analysis on Semisimple Lie Groups*, Mathematical Surveys and Monographs, Vol. 31, Amer. Math. Soc. (1989) 223–286.

[St] N. Steenrod, *The Topology of Fibre Bundles* (Princeton University Press, Princeton, NJ, 1951).

[V] D.A. Vogan, Jr., Unipotent representations and cohomological induction, Contemporary Math. 154 (1993) 47–70.

[WW] R.O. Wells, Jr. and J. Wolf, Poincaré series and automorphic cohomology on flag domains, Ann. Math. 105 (1977) 397–448.

[W] J. Wolf, The Stein condition for cycle spaces of open orbits on complex flag manifolds, Ann. Math. 136 (1992) 541–555.

[Wo] H.W. Wong, Dolbeault cohomologies and Zuckerman modules associated with finite rank representations, Ph.D. Dissertation, Harvard Univ. (1992).